Acoustic energy and momentum in a moving medium

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By exploiting the mathematical analogy between the propagation of sound in a nonhomogeneous potential flow and the propagation of a scalar field in curved space-time, various wave ''energy'' and wave ''momentum'' conservation laws are established in a systematic manner. In particular, the acoustic energy conservation law due to Blokhintsev appears as the result of the conservation of a mixed covariant and contravariant energy-momentum tensor, while the exchange of relative energy between the wave and mean flow, first noted by Longuet-Higgins and Stewart in the context of ocean waves, appears as the covariant conservation of the doubly contravariant form of the same energy-momentum tensor.

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I. INTRODUCTION

Many discussions of the ''energy'' and ''momentum'' associated with waves propagating through moving fluids can be found in the physics [1], engineering [2–6], and mathematical fluid mechanics literature [7–16]. Various definitions are proposed, some of which lead to conserved quantities, and some to quantities that are not conserved but instead exchanged between the wave and the mean flow. In part, the multiplicity of definitions is due to difficulty in deciding what fraction of the energy or momentum of the system properly belongs to the wave and what fraction should be associated with the moving medium. It is also often unclear how to divide equations expressing conservation laws into terms relating to the conserved quantity, and terms acting as sources for this quantity. Related to these primarily cosmetic problems are more fundamental issues as to whether the ''energy'' or ''momentum'' under discussion is the true Newtonian energy or momentum, or instead pseudoenergy and pseudomomentum. Thus we have the question ''What is the momentum of a sound wave?'' raised by Peierls in his book *Surprises in Theoretical Physics* [17], and the salutary polemic ''On the Wave Momentum Myth'' by McIntyre [18].

The most extensive analyses of conserved wave properties have been carried out by the fluid mechanics community [7–15]. Typically these papers adopt a Lagrangian (following individual particles in the flow) or mixed Lagrangian-Eulerian approach, as opposed to the purely Eulerian (describing the flow in terms of a velocity field) approach which would be most familiar to a physicist. In addition, a physicist reading this literature feels the lack of a general organizing principle behind the definition and derivation of the conservation laws. The present paper is intended to remedy some of these problems—at least for the special case of sound waves propagating through an irrotational homentropic flow. Although a rather restricted class of motions, this is still one of considerable interest in condensed matter physics as it includes phonon propagation in a Bose condensate, and so lies at the heart of the two-fluid model of superfluidity. By exploiting Unruh's ingenious identification [19,20] of the wave equation for sound waves in such a flow with the equation for a scalar field propagating in curved space-time, I extract the conservation laws from the principle of general covariance. Deriving the conservation laws in this way may seem like a case of taking a sledgehammer to crack a nut, but the formalism is familar to most physicists, automatic in application, and the ambiguities in defining the conserved quantities turn out to lie in the choice of whether to identify the energy-momentum tensor as $T^{\mu\nu}$ or as T^{μ}_{ν}. Also, when quantities are not conserved, as is the case of the wave momentum in a shear flow, their sources arise naturally from the connection terms in the covariant derivative.

In Sec. II, I discuss the action describing the irrotational motion of a homentropic fluid. In Sec. III, I derive Unruh's equation from the action principle. In Sec. IV, I explain why we often need information beyond the solutions of the linearized wave equation, and in Sec. V derive the conservation equations that follow from the linearized equation. Section VI interprets these equations in terms of the motion of phonons. In the discussion section I consider the connection between the conservation laws and forces.

The work reported here was motivated by a desire to better understand the role of acoustic radiation stress in the two-fluid model. It may be relevent to the recent controversy [22–25] over the *Iordanskii force* acting on a vortex moving with respect to the normal fluid component. The use of the Unruh formalism in this context was suggested by Volovik [26].

II. THE ACTION PRINCIPLE

The most straightforward way of deriving conservation laws starts with an action principle. From this, Noether's theorem provides us with an explicit formula for a conserved quantity corresponding to each symmetry of the action. In fluid mechanics unfortunately—at least when we restrict ourselves to a eulerian description of the flow field—action principles are in short supply. Of course there must exist *some* action principle because ultimately the fluid can be treated as a system of particles. A particle-based action, however, requires a Lagrangian description of the flow. When it is reex-

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pressed in Eulerian terms, constraints appear, and these limit its utility.

If we restrict ourselves to flows that are both irrotational and homentropic—the latter term meaning in practice that pressure is a function of the fluid density only—then the number of degrees of freedom available to the fluid is dramatically reduced. In this case the Eulerian equations of motion *are* derivable from the action [27]

$$S=\int d^4x[\rho\dot{\phi}+\tfrac{1}{2}\rho(\nabla\phi)^2+u(\rho)]. \tag{2.1}$$

Here ρ is the mass density, ϕ the velocity potential, and the overdot denotes differentiation with respect to time. The function u may be identified with the internal energy density.

Equating to zero the variation of S with respect to ϕ yields the continuity equation

$$\dot{\rho}+\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0, \tag{2.2}$$

where $\mathbf{v}\equiv\boldsymbol{\nabla}\phi$. Varying ρ gives Bernoulli's equation

$$\dot{\phi}+\tfrac{1}{2}\mathbf{v}^2+\mu(\rho)=0, \tag{2.3}$$

where $\mu(\rho)=du/d\rho$. In most applications μ would be identified with the specific enthalpy. For a superfluid condensate the entropy density s is identically zero and μ is the local chemical potential.

It is worth noting that our action could *not* have arisen from some rewriting of the action for the motion of a system of individual particles. We are allowing variations of ρ without requiring simultaneous variations of ϕ, and such variations conjure new matter out of nothing.

The gradient of the Bernoulli equation is Euler's equation of motion for the fluid. Combining this with the continuity equation yields a momentum conservation law

$$\partial_t(\rho v_i)+\partial_j(\rho v_j v_i)+\rho\partial_i\mu=0. \tag{2.4}$$

We simplify Eq. (2.4) by introducing the pressure P, which is related to μ by $P(\rho)=\int\rho\, d\mu$. Then we can write

$$\partial_t(\rho v_i)+\partial_j\Pi_{ji}=0, \tag{2.5}$$

where Π_{ij} is given by

$$\Pi_{ij}=\rho v_i v_j+\delta_{ij}P. \tag{2.6}$$

This is the usual form of the momentum flux tensor in fluid mechanics.

The relations $\mu=du/d\rho$ and $\rho=dP/d\mu$ show that P and u are related by a Legendre transformation: $P=\rho\mu-u(\rho)$. From this and the Bernoulli equation we see that the pressure is equal to minus the action density:

$$-P=\rho\dot{\phi}+\tfrac{1}{2}\rho(\nabla\phi)^2+u(\rho). \tag{2.7}$$

Consequently, we can write

$$\Pi_{ij}=\rho\partial_i\phi\partial_j\phi-\delta_{ij}[\rho\dot{\phi}+\tfrac{1}{2}\rho(\nabla\phi)^2+u(\rho)]. \tag{2.8}$$

This is the flux tensor that would appear were we to use Noether's theorem to derive a law of momentum conservation directly from the invariance of the action under the translation $\phi(\mathbf{r})\to\phi(\mathbf{r}-\mathbf{a})$, $\rho(\mathbf{r})\to\rho(\mathbf{r}-\mathbf{a})$. This is not a trivial point because there are two similar, but distinct, notions of "momentum." True momentum is associated with the symmetry of the action under a simultaneous translation of all the particles in the system. Its conservation requires an absence of external forces. *Pseudomomentum* [21] is the quantity that is preserved when the action is left invariant when the *disturbance* in the medium is relocated, but the reference position of each individual particle is left unchanged. Conservation of pseudomomentum requires homogeneity of the medium rather than of space. Replacing the field $\phi(\mathbf{r})$ by $\phi(\mathbf{r}-\mathbf{a})$ would normally correspond to the latter symmetry, but, because of the absence of explicit particles, at this point in our discussion the two concepts coincide.

III. THE UNRUH METRIC

We now obtain the linearized wave equation for the propagation of sound waves in a background mean flow. Let

$$\begin{aligned}\phi&=\phi_{(0)}+\phi_{(1)},\\ \rho&=\rho_{(0)}+\rho_{(1)}.\end{aligned} \tag{3.1}$$

Here $\phi_{(0)}$ and $\rho_{(0)}$ define the mean flow. We assume that they obey the equations of motion. The quantities $\phi_{(1)}$ and $\rho_{(1)}$ represent small amplitude perturbations. Expanding S to quadratic order in these perturbations gives

$$\begin{aligned}S=S_0+\int d^4x\bigg[&\rho_{(1)}\dot{\phi}_{(1)}+\frac{1}{2}\left(\frac{c^2}{\rho_{(0)}}\right)\rho_{(1)}^2\\ &+\frac{1}{2}\rho_{(0)}(\nabla\phi_{(1)})^2+\rho_{(1)}\mathbf{v}\cdot\nabla\phi_{(1)}\bigg].\end{aligned} \tag{3.2}$$

Here $\mathbf{v}\equiv\mathbf{v}_{(0)}=\nabla\phi_{(0)}$. The speed of sound c is defined by

$$\frac{c^2}{\rho_{(0)}}=\left.\frac{d\mu}{d\rho}\right|_{\rho_{(0)}}, \tag{3.3}$$

or more familiarly

$$c^2=\frac{dP}{d\rho}. \tag{3.4}$$

The terms linear in the perturbations vanish because of our assumption that the zeroth-order variables obey the equation of motion.

The equation of motion for $\rho_{(1)}$ derived from Eq. (3.2) is

$$\rho_{(1)}=-\frac{\rho_{(0)}}{c^2}(\dot{\phi}_{(1)}+\mathbf{v}\cdot\nabla\phi_{(1)}). \tag{3.5}$$

In general we are not allowed to substitute a consequence of an equation of motion back into the action integral. Here, however, because $\rho_{(1)}$ occurs quadratically, we may use Eq. (3.5) to eliminate it and obtain an effective action for the potential $\phi_{(1)}$ only,

$$S_{(2)}=\int d^4x\left(\frac{1}{2}\rho_{(0)}(\nabla\phi_{(1)})^2-\frac{\rho_{(0)}}{2c^2}(\dot{\phi}_{(1)}+\mathbf{v}\cdot\nabla\phi_{(1)})^2\right). \tag{3.6}$$

The resultant equation of motion for $\phi_{(1)}$ is [19,20]

$$\left(\frac{\partial}{\partial t}+\nabla\cdot\mathbf{v}\right)\frac{\rho_{(0)}}{c^2}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\phi_{(1)}=\nabla(\rho_{(0)}\nabla\phi_{(1)}). \tag{3.7}$$

Note that in deriving this equation we have *not* assumed that the background flow $\mathbf{v}$ is steady, only that it satisfies the equations of motion. Naturally, in order for our waves to be distinguishable from the background flow, the latter should be slowly changing and have a longer length scale than the wave motion.

Equation (3.7) can be rewritten so as to contain convective time derivatives:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\phi_{(1)}=\frac{1}{\rho_{(0)}}\nabla(\rho_{(0)}\nabla\phi_{(1)}). \tag{3.8}$$

The equivalence of Eqs. (3.8) and (3.7) is established by using the mass conservation equation $\partial_t\rho_{(0)}+\nabla\cdot\rho_{(0)}\mathbf{v}=0$. At this point it is worth noting that an equation having the appearance of Eq. (3.8) was derived by Pierce [28] without any restriction to irrotational motion—but only as an approximation valid for slowly varying background flows. In Pierce's derivation, the dependent variable, which he calls Φ, is no longer exactly the velocity potential, and the relation $v_{(1)}=\nabla\Phi$ has corrections of $O(L^{-1})+O(T^{-1})$, where L and T are the characteristic length and time of the background flow inhomogeneities.

Although Eq. (3.8) may seem more familiar, the form (3.7) has the advantage that it can be written as[1]

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi_{(1)}=0, \tag{3.9}$$

where

$$\sqrt{-g}g^{\mu\nu}=\frac{\rho_{(0)}}{c^2}\begin{pmatrix}1 & \mathbf{v}^T\\ \mathbf{v} & \mathbf{v}\mathbf{v}^T-c^2\mathbf{1}\end{pmatrix}. \tag{3.10}$$

This is perhaps most easily seen by observing that the action (3.6) is equal to $-S$ where

$$S=\int d^4x\tfrac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi_{(1)}\partial_\nu\phi_{(1)}=\int d^4x\sqrt{-g}L. \tag{3.11}$$

Equation (3.9) has the same form as that of a scalar wave propagating in a gravitational field with Riemann metric $g_{\mu\nu}$. The idea of writing the acoustic wave equation in this way is due to Unruh [19,20]. I will therefore refer to $g_{\mu\nu}$ as the Unruh metric.

As is customary in general relativity, the symbol g denotes the determinant of the covariant form of the metric, $g_{\mu\nu}$, so $\det g^{\mu\nu}=g^{-1}$. Taking the determinant of both sides of Eq. (3.10) thus shows that the four-volume measure $\sqrt{-g}$ is equal to $\rho_{(0)}^2/c$. Knowing this, we may then invert the matrix $g^{\mu\nu}$ to find the covariant components of the metric,

$$g_{\mu\nu}=\frac{\rho_{(0)}}{c}\begin{pmatrix}c^2-v^2 & \mathbf{v}^T\\ \mathbf{v} & -\mathbf{1}\end{pmatrix}. \tag{3.12}$$

The associated space-time interval is therefore

$$ds^2=\frac{\rho_{(0)}}{c}[c^2dt^2-\delta_{ij}(dx^i-v^idt)(dx^j-v^jdt)]. \tag{3.13}$$

Metrics of this form, although without the overall conformal factor $\rho_{(0)}/c$, appear in the Arnowitt-Deser-Misner (ADM) formalism of general relativity [29]. There, c and $-v^i$ are refered to as the *lapse function* and *shift vector*, repectively. They serve to glue successive three-dimensional time slices together to form a four-dimensional space-time [30]. In our present case, provided $\rho_{(0)}/c$ can be regarded as a constant, each three-space is ordinary flat $\mathbf{R}^3$ equipped with the rectangular Cartesian metric $g_{ij}^{(space)}=\delta_{ij}$—but the resultant space-time is in general curved, the curvature depending on the degree of inhomogeneity of the mean flow $\mathbf{v}$.

In the geometric acoustics limit, sound will travel along the null geodesics defined by $g_{\mu\nu}$. Even in the presence of spatially varying $\rho_{(0)}$ we would expect the ray paths to depend only on the local values of c and $\mathbf{v}$, so it is perhaps a bit surprising to see the density entering the expression for the Unruh metric. An overall conformal factor, however, does not affect *null* geodesics, and thus variations in $\rho_{(0)}$ do not influence the ray tracing. For steady flow, and in the case that only $\mathbf{v}$ is varying, it is shown in the Appendix that the null geodesics coincide with the ray paths obtained by applying Hamilton's equations for rays,

$$\dot{x}^i=\frac{\partial\omega}{\partial k_i},\quad \dot{k}_i=-\frac{\partial\omega}{\partial x^i}, \tag{3.14}$$

to the appropriate Doppler shifted frequency

$$\omega(\mathbf{x},\mathbf{k})=c|\mathbf{k}|+\mathbf{v}\cdot\mathbf{k}. \tag{3.15}$$

When $\mathbf{v}$ lies in the x direction only, we can also rewrite ds^2 as

$$ds^2=\frac{\rho_{(0)}}{c}\{-[dx-(v+c)dt][dx-(v-c)dt]-dy^2-dz^2\}. \tag{3.16}$$

This shows that the x-t plane null geodesics coincide with the expected characteristics of the wave equation in the background flow.

[1] I use the convention that greek letters run over four space-time indices 0,1,2,3 with $0\equiv t$, while roman indices refer to the three space components.

IV. SECOND-ORDER QUANTITIES

The fluid in a sound wave has average velocity zero, but, since the fluid is compressed in the half cycle when it is moving in the direction of propagation and rarefied when it is moving backward, there is a net mass current (and hence a momentum density) which is of second order in the sound wave amplitude a_0. This becomes clearer if one solves the equation

$$\frac{d\xi}{dt}=v(\xi)=a_0\cos(k\xi-\omega t) \tag{4.1}$$

for the trajectory $x=\xi(t)$ of a fluid particle. This equation is nonlinear and, solving perturbatively, one finds a secular drift at second order in a_0:

$$\xi(t)=\xi(0)+\text{oscillations}+\frac{1}{2}a_0^2\left(\frac{k}{\omega}\right)t+\cdots. \tag{4.2}$$

Although the time average of the Eulerian fluid velocity v is zero, the time average of the *Lagrangian* velocity $v_L=\dot{\xi}$ is not. The difference beweeen the two average velocities is the *Stokes drift*. The Stokes drift is $O(a_0^2)$ while the wave equation is accurate only to $O(a_0)$, so care is necessary before using its solutions to evaluate the mass current. Similar problems occur in defining the energy density and energy and momentum fluxes, which also require second-order accuracy.

We can expand the velocity field as

$$\mathbf{v}=\mathbf{v}+\mathbf{v}_{(1)}+\mathbf{v}_{(2)}+\cdots, \tag{4.3}$$

where the second-order correction $\mathbf{v}_{(2)}$ arises as as consequence of the nonlinearities in the equations of motion. This correction will possess both oscillating and steady components. The oscillatory components arise because a strictly harmonic wave with frequency ω_0 will gradually develop higher frequency components due to the progressive distortion of the wave as it propagates. (A plane wave eventually degenerates into a sequence of shocks.) These distortions are usually not significant in considerations of energy and momentum balance. The steady terms, however, represent $O(a_0^2)$ alterations to the mean flow caused by the sound waves, and these often possess energy and momentum comparable to that of the sound field.

Even if we temporarily ignore these effects and retain only $\mathbf{v}_{(1)}$ as determined from the linearized wave equation, the density and pressure will still have expansions

$$\begin{aligned}\rho&=\rho_{(0)}+\rho_{(1)}+\rho_{(2)}+\cdots,\\ P&=P_{(0)}+P_{(1)}+P_{(2)}+\cdots.\end{aligned} \tag{4.4}$$

As before, the grading (n) refers to the number of powers of the sound wave amplitude in an expression. The small parameter in these expansions is the Mach number given by a typical value of $v_{(1)}$ divided by the local speed of sound.

Consider, for example, the momentum density $\rho\mathbf{v}$ and the momentum flux

$$\Pi_{ij}=\rho v_i v_j+\delta_{ij}P. \tag{4.5}$$

It is reasonable to define the momentum density and the momentum flux tensor associated with the sound field as the second-order terms

$$\mathbf{j}^{(\text{phonon})}=\langle\rho_{(1)}\mathbf{v}_{(1)}\rangle+\mathbf{v}\langle\rho_{(2)}\rangle, \tag{4.6}$$

and

$$\begin{aligned}\Pi_{ij}^{(\text{phonon})}=&\rho_{(0)}\langle v_{(1)i}v_{(1)j}\rangle+v_i\langle\rho_{(1)}v_{(1)j}\rangle+v_j\langle\rho_{(1)}v_{(1)i}\rangle\\ &+\delta_{ij}\langle P_{(2)}\rangle+v_iv_j\langle\rho_{(2)}\rangle.\end{aligned} \tag{4.7}$$

(The angular brackets indicate that we should take a time average over a sound wave period. There is no need to consider terms first order in the amplitude because these average to zero.) We see that we need to consider the second-order contributions to both P and ρ.

We can compute $P_{(2)}$ in terms of first-order quantities from

$$\Delta P=\frac{dP}{d\mu}\Delta\mu+\frac{1}{2}\frac{d^2P}{d\mu^2}(\Delta\mu)^2+O((\Delta\mu)^3) \tag{4.8}$$

and Bernoulli's equation in the form

$$\Delta\mu=-\dot{\phi}_{(1)}-\tfrac{1}{2}(\nabla\phi_{(1)})^2-\mathbf{v}\cdot\boldsymbol{\nabla}\phi_{(1)}, \tag{4.9}$$

together with

$$\frac{dP}{d\mu}=\rho,\quad \frac{d^2P}{d\mu^2}=\frac{d\rho}{d\mu}=\frac{\rho}{c^2}. \tag{4.10}$$

Expanding out and grouping terms of appropriate orders gives

$$P_{(1)}=-\rho_{(0)}(\dot{\phi}_{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla}\phi_{(1)})=c^2\rho_{(1)}, \tag{4.11}$$

which we already knew, and

$$P_{(2)}=-\rho_{(0)}\tfrac{1}{2}(\nabla\phi_{(1)})^2+\frac{1}{2}\frac{\rho_{(0)}}{c^2}(\dot{\phi}_{(1)}+\mathbf{v}\cdot\boldsymbol{\nabla}\phi_{(1)})^2. \tag{4.12}$$

We see that $P_{(2)}=\sqrt{-g}L$ where L is the Lagrangian density for our sound wave equation.

To extract $\rho_{(2)}$ in this manner we need more information about the equation of state of the fluid than is used in the linearized theory. This information is most conveniently parametrized by the logarithmic derivative of the speed of sound with pressure (a fluid-state analog of the Grüneisen parameter). Using this together with the previous results for $P_{(2)}$, we find that

$$\rho_{(2)}=\frac{1}{c^2}P_{(2)}-\frac{1}{\rho_{(0)}}\rho_{(1)}^2\left.\frac{d\ln c}{d\ln\rho}\right|_{\rho_{(0)}}. \tag{4.13}$$

V. CONSERVATION LAWS

While we cannot compute the "true" energy and momentum densities and fluxes without including nonlinear corrections to the motion, it is often more useful to find closely related quantities whose conservation laws are a consequence of the linearized wave equation, and therefore provide information about the solutions of this equation. Our ''general relativistic'' formalism provides a sytematic way of finding such conserved quantities. It is well known [31] that any action S automatically provides us with a covariantly conserved and symmetric energy-momentum tensor $T_{\mu\nu}$ defined by

$$T_{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{5.1}$$

The functional derivative is here defined by

$$\delta S=\int d^4x\sqrt{-g}\frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu}. \tag{5.2}$$

It follows from the equations of motion derived from S that

$$D_\mu T^{\mu\nu}=0, \tag{5.3}$$

where D_μ is the covariant derivative. For example,

$$D_\alpha A^{\mu\nu}_\sigma=\partial_\alpha A^{\mu\nu}_\sigma+\Gamma^\mu_{\alpha\gamma}A^{\gamma\nu}_\sigma+\Gamma^\nu_{\alpha\gamma}A^{\mu\gamma}_\sigma-\Gamma^\gamma_{\alpha\sigma}A^{\mu\nu}_\gamma. \tag{5.4}$$

The $\Gamma^\alpha_{\beta\gamma}$ are the components of the Levi-Civitá connection compatable with the Unruh metric, viz.,

$$\Gamma^\alpha_{\beta\gamma}=g^{\alpha\rho}[\beta\gamma,\rho], \tag{5.5}$$

where

$$[\beta\gamma,\rho]=\frac{1}{2}\left(\frac{\partial g_{\gamma\rho}}{\partial x^\beta}+\frac{\partial g_{\beta\rho}}{\partial x^\gamma}-\frac{\partial g_{\beta\gamma}}{\partial x^\rho}\right). \tag{5.6}$$

For our scalar field

$$T^{\mu\nu}=\partial^\mu\phi_{(1)}\partial^\nu\phi_{(1)}-g^{\mu\nu}(\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\phi_{(1)}\partial_\beta\phi_{(1)}). \tag{5.7}$$

The derivatives with raised indices in Eq. (5.7) are defined by

$$\partial^0\phi_{(1)}=g^{0\mu}\partial_\mu\phi_{(1)}=\frac{1}{\rho_{(0)}c}(\dot\phi_{(1)}+\mathbf{v}\cdot\nabla\phi_{(1)}) \tag{5.8}$$

and

$$\partial^i\phi_{(1)}=g^{i\mu}\partial_\mu\phi_{(1)}=\frac{1}{\rho_{(0)}c}[v_i(\dot\phi_{(1)}+\mathbf{v}\cdot\nabla\phi_{(1)})-c^2\partial_i\phi_{(1)}]. \tag{5.9}$$

Thus

$$\begin{aligned}T^{00}&=\frac{1}{\rho^3_{(0)}}\left(\rho_{(0)}\frac{1}{2}(\nabla\phi_{(1)})^2+\frac{1}{2}\frac{\rho_{(0)}}{c^2}(\dot\phi_{(1)}+\mathbf{v}\cdot\nabla\phi_{(1)})^2\right)\\&=\frac{c^2}{\rho^3_{(0)}}\left(\frac{W_r}{c^2}\right)\\&=\frac{c^2}{\rho^3_{(0)}}\tilde\rho_{(2)}.\end{aligned} \tag{5.10}$$

The last two equalities serve as a definition of W_r and $\tilde\rho_{(2)}$. The quantity W_r is often decribed as the acoustic energy density relative to the frame moving with the local fluid velocity [11]. Because its conservation requires a steady flow, rather than the absence of time-dependent external forces, it is more correctly a pseudoenergy density.

Using Eq. (4.11), and Eq. (4.12) in the form

$$\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\phi_{(1)}\partial_\beta\phi_{(1)}=\frac{c}{\rho^2_{(0)}}P_{(2)}, \tag{5.11}$$

we can express the other components of Eq. (5.7) in terms of physical quantities. We find that

$$\begin{aligned}T^{i0}&=T^{0i}\\&=\frac{c^2}{\rho^3_{(0)}}\left(\frac{1}{c^2}(P_{(1)}v_{(1)i}+v_iW_r)\right)\\&=\frac{c^2}{\rho^3_{(0)}}(\rho_{(1)}v_{(1)i}+v_i\tilde\rho_{(2)}).\end{aligned} \tag{5.12}$$

The first line in this expression shows that, up to an overall factor, T^{i0} is the energy flux—the first term being the rate of working by a fluid element on its neigbour, and the second the advected energy. The second line is written so as to suggest the usual relativistic identification of (energy flux)/c^2 with the density of momentum. This interpretation, however, requires that $\tilde\rho_{(2)}$ be the second-order correction to the density, which it is not.

Similarly,

$$\begin{aligned}T^{ij}=\frac{c^2}{\rho^3_{(0)}}(&\rho_{(0)}v_{(1)i}v_{(1)j}+v_i\rho_{(1)}v_{(1)j}\\&+v_j\rho_{(1)}v_{(1)i}+\delta_{ij}P_{(2)}+v_iv_j\tilde\rho_{(2)}).\end{aligned} \tag{5.13}$$

We again see that if we identify $\tilde\rho_{(2)}$ with $\rho_{(2)}$ then T^{ij} has the exactly the form we expect for the second-order momentum flux tensor.

The reason why the linear theory makes the erroneous identification of $\rho_{(2)}$ with $\tilde\rho_{(2)}$ is best seen if we set $\mathbf{v}$ $=$const. Then the equation

$$\partial_t T^{00}+\partial_i T^{i0}=0 \tag{5.14}$$

holds. This reads

$$\frac{c^2}{\rho^3_{(0)}}[\partial_t\tilde\rho_{(2)}+\partial_i(\rho_{(1)}v_{(1)i}+v_i\tilde\rho_{(2)})]=0. \tag{5.15}$$

This looks very much like the second-order continuity equation

$$\partial_t\rho_{(2)}+\partial_i(v_{(2)}\rho_{(0)}+\rho_{(1)}v_{(1)i}+v_i\rho_{(2)})=0, \tag{5.16}$$

since the linear theory ignores $\mathbf{v}_{(2)}$. When we retain its $\mathbf{v}_{(2)}$ term, however, Eq. (5.16) ceases to be an equation determining $\rho_{(2)}$, and instead, after time averaging, shows that $\nabla\cdot\langle\rho_{(0)}\mathbf{v}_{(2)}\rangle\neq 0$ in an inhomogeneous sound field [32].

We can also write the mixed covariant and contravariant components of the energy momentum tensor $T^{\mu}_{\nu}=T^{\mu\lambda}g_{\lambda\nu}$ in terms of physical quantities. This mixed tensor turns out to be more useful than the doubly contravariant tensor. Because we no longer enforce a symmetry between the indices μ and ν, the quantity W_r is no longer required to perform double duty as both an energy and a density. We find

$$T^0_0=\frac{c}{\rho^2_{(0)}}(W_r+\rho_{(1)}\mathbf{v}_{(1)}\cdot\mathbf{v}),$$

$$T^i_0=\frac{c}{\rho^2_{(0)}}\left(\frac{P_{(1)}}{\rho_{(0)}}+\mathbf{v}\cdot\mathbf{v}_{(1)}\right)(\rho_{(0)}v_{(1)i}+\rho_{(1)}v_{(0)i}) \quad (5.17)$$

and

$$T^0_i=-\frac{c}{\rho^2_{(0)}}\rho_{(1)}\mathbf{v}_{(1)i},$$

$$T^i_j=-\frac{c}{\rho^2_{(0)}}(\rho_{(0)}v_{(1)i}v_{(1)j}+v_i\rho_{(1)}v_{(1)j}+\delta_{ij}P_{(2)}). \quad (5.18)$$

We see that $\tilde{\rho}_{(2)}$ does not appear here, and all these terms may be identified with physical quantities that are reliably computed from solutions of the linearized wave equation.

The covariant conservation law can be written either $D_\mu T^{\mu\nu}=0$ or $D_\mu T^{\mu}_{\nu}=0$. The two equations are consistent with each other because the covariant derivative is defined so that $D_\lambda g_{\mu\nu}=g_{\mu\nu}D_\lambda$. To extract the physical meaning of these equations we need to evaluate the the connection forms $\Gamma^{\mu}_{\nu\lambda}$.

In what follows I will consider only a steady background flow, and, further, one for which ρ_0, c, and hence $\sqrt{-g}=\rho^2_{(0)}/c$ can be treated as constant. To increase the readability of some expressions I will also choose units so that ρ_0 and c become unity and no longer appear as overall factors in the metric or the four-dimensional energy-momentum tensors. I will, however, reintroduce them when they are required for dimensional correctness in expressions such as $\rho_{(0)}\mathbf{v}_{(1)}$ or W_r/c^2.

From the Unruh metric we find

$$\begin{aligned}
[ij,k]&=0,\\
[ij,0]&=\tfrac{1}{2}(\partial_i v_j+\partial_j v_i),\\
[i0,j]&=\tfrac{1}{2}(\partial_i v_j-\partial_j v_i),\\
[0i,0]&=[i0,0]=-\tfrac{1}{2}\partial_i|v|^2\\
[00,i]&=\tfrac{1}{2}\partial_i|v|^2,\\
[00,0]&=0.
\end{aligned} \quad (5.19)$$

I have retained the expression $\frac{1}{2}(\partial_i v_j-\partial_j v_i)$ in $[i0,j]$, since the previously cited paper by Pierce [28] indicates that our wave equation also applies to weakly inhomogeneous rotational flows.

We therefore find

$$\begin{aligned}
\Gamma^0_{00}&=\tfrac{1}{2}(\mathbf{v}\cdot\nabla)|v|^2,\\
\Gamma^0_{i0}&=-\tfrac{1}{2}\partial_i|v|^2+\tfrac{1}{2}v_j(\partial_i v_j-\partial_j v_i),\\
\Gamma^i_{00}&=\tfrac{1}{2}v_i(\mathbf{v}\cdot\nabla)|v|^2-\tfrac{1}{2}\partial_i|v|^2,\\
\Gamma^0_{ij}&=\tfrac{1}{2}(\partial_i v_j+\partial_j v_i),\\
\Gamma^i_{j0}&=-\tfrac{1}{2}v_i\partial_j|v|^2+\tfrac{1}{2}(\partial_j v_k-\partial_k v_j)(v_k v_i-c^2\delta_{ik}),\\
\Gamma^i_{jk}&=\tfrac{1}{2}v_i(\partial_j v_k+\partial_k v_j).
\end{aligned} \quad (5.20)$$

From Eq. (5.6) we have

$$\Gamma^{\mu}_{\mu\beta}=\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^\beta}, \quad (5.21)$$

so, with $\sqrt{-g}=\text{const.}$, the trace $\Gamma^{\mu}_{\mu\beta}$ is zero. One may verify that the above expressions for $\Gamma^{\mu}_{\nu\lambda}$ obey this identity.

We now evaluate

$$\begin{aligned}
D_\mu T^{\mu 0}&=\partial_\mu T^{\mu 0}+\Gamma^{\mu}_{\mu\gamma}T^{\gamma 0}+\Gamma^0_{\mu\nu}T^{\mu\nu}\\
&=\partial_\mu T^{\mu 0}+\Gamma^0_{\mu\nu}T^{\mu\nu}.
\end{aligned} \quad (5.22)$$

After a little algebra we find

$$\Gamma^0_{\mu\nu}T^{\mu\nu}=\tfrac{1}{2}(\partial_i v_j+\partial_j v_i)(\rho_{(0)}v_{(1)i}v_{(1)j}+\delta_{ij}P_{(2)}). \quad (5.23)$$

Note the nonappearence of $\rho_{(1)}$ and $\tilde{\rho}_{(2)}$ in the final expression—even though both quantities appear in $T^{\mu\nu}$.

The conservation law therefore becomes

$$\partial_t W_r+\partial_i(P_{(1)}v_{(1)i}+v_i W_r)+\frac{1}{2}\Sigma_{ij}(\partial_i v_j+\partial_j v_i)=0, \quad (5.24)$$

where

$$\Sigma_{ij}=\rho_{(0)}v_{(1)i}v_{(1)j}+\delta_{ij}P_{(2)}. \quad (5.25)$$

This is an example of the general form of energy law derived by Longuet-Higgins and Stuart, originally in the context of ocean waves [14,15]. (See also [4] for a slightly earlier, but less general, case.) The relative energy density $W_r\equiv T^{00}$ is *not* conserved. Instead, an observer moving with the fluid sees the waves acquiring energy from the mean flow at a rate given by the product of a radiation stress Σ_{ij} with the mean-flow rate of strain. Such nonconservation is not surprising. Seen from the viewpoint of the moving frame, the mean flow is no longer steady, and (pseudo)energy conservation requires a time-independent medium.

Note that, since we are assuming that $\rho_{(0)}$ is a constant, we should for consistency require $\nabla\cdot\mathbf{v}=0$. Thus the *isotropic* part of the radiation stress (the part proportional to δ_{ij}) does no work. This is fortunate because the nonlinear theory shows that the isotropic radiation stress contains a part dependent on $d\ln c/d\ln\rho$ that is missed by the linear approximation. (See, however, [33].)

We now examine the energy conservation law coming from the zeroth component of the mixed energy-momentum tensor. We need

$$D_\mu T_0^\mu = \partial_\mu T_0^\mu - \Gamma^\rho_{\mu 0} T^\mu_\rho$$
$$= \partial_\mu T_0^\mu - [\mu 0,\rho] T^{\mu\rho}$$
$$= \partial_\mu T_0^\mu - [i0,0] T^{i0} - [00,i] T^{0i} - [i0,j] T^{ij}. \tag{5.26}$$

We now observe that $T^{i0}=T^{0i}$ while $[00,i]=-[i0,0]$, and that $[i0,j]=-[j0,i]$, while $T^{ij}=T^{ji}$. Thus the connection contribution vanishes. This form of the energy conservation law is therefore

$$\partial_t(W_r+\rho_{(1)}\mathbf{v}_{(1)}\cdot\mathbf{v})+\partial_i\left[\left(\frac{P_{(1)}}{\rho_{(0)}}+\mathbf{v}\cdot\mathbf{v}_{(1)}\right)\times(\rho_{(0)}v_{(1)i}+\rho_{(1)}v_{(0)i})\right]=0. \tag{5.27}$$

Here we see that the combination $W_r+\rho_{(1)}\mathbf{v}_{(1)}\cdot\mathbf{v}$ *does* correspond to a conserved energy. This conservation law was originally derived by Blokhintsev [2] for slowly varying flows, and more generally by Cantrell and Hart [3] in their study of the acoustic stability of rocket engines. [See also Refs. [5] and [13] Eq. (5.18).]

Now we turn to the equation for momentum conservation. Working as for the energy law we find

$$D_\mu T_j^\mu = \partial_\mu T_j^\mu - [\mu j,\rho] T^{\mu\rho}$$
$$= \partial_\mu T_j^\mu - [0j,0]T^{00} - [ij,0]T^{i0} - [0j,i]T^{0i}$$
$$= \partial_\mu T_j^\mu - \rho_{(1)} v_{(1)i}\partial_j v_i . \tag{5.28}$$

Again note the cancellation of the terms containing $\tilde{\rho}_{(2)}$.

The covariant conservation equation $D_\mu T_j^\mu = 0$ therefore reads

$$\partial_t \rho_{(1)} v_{(1)j} + \partial_i(\rho_{(0)} v_{(1)i} v_{(1)j} + v_i \rho_{(1)} v_{(1)j} + \delta_{ij} P_{(2)}) + \rho_{(1)} v_{(1)i}\partial_j v_i = 0. \tag{5.29}$$

The connection terms have provided a source term for the momentum density. Thus, in an inhomogeneous flow field, momentum is exchanged beween the waves and the mean flow.

If we accept that our wave equation continues to be valid for weakly inhomogeneous rotational flows, then from Eq. (5.29) we can derive an expression for the time average of the divergence of the radiation stress tensor:

$$\langle \partial_i(\rho_{(0)} v_{(1)i} v_{(1)j} + v_i\rho_{(1)} v_{(1)j} + \rho_{(1)} v_{(1)i} v_j + P_{(2)}\delta_{ij})\rangle = -\langle \rho_{(1)} v_{(1)i}\rangle(\partial_j v_i - \partial_i v_j) + v_j\langle \partial_i \rho_{(1)} v_{(1)i}\rangle. \tag{5.30}$$

The sound wave therefore exerts a body force on the background flow, one part of which is analogous to the Lorentz force, the role of the magnetic field being played by the vorticity. The relation between this body force and the Iordanskii force on a line vortex is the same as that between the conventional Lorentz force and the Aharonov-Bohm force on a narrow flux tube [24,25].

VI. PHONONS AND CONSERVATION OF WAVE ACTION

If the mean flow varies slowly on the scale of a wavelength, the sound field can locally be approximated by a plane wave,

$$\phi(x,t)=a_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t). \tag{6.1}$$

The frequency ω and the wave vector $\mathbf{k}$ are here related by the Doppler-shifted dispersion relation $\omega=\omega_r+\mathbf{k}\cdot\mathbf{v}$, where the relative frequency $\omega_r=c|k|$ is that measured in the frame moving with the fluid. A packet of such waves moves at the group velocity

$$\mathbf{U}=\dot{\mathbf{x}}=c\frac{\mathbf{k}}{|k|}+\mathbf{v}. \tag{6.2}$$

As the wave progresses through regions of varying $\mathbf{v}$, the parameters $\mathbf{k}$ and a_0 will slowly evolve. The change in $\mathbf{k}$ is given by the ray tracing formula [Eq. (A16)]

$$\frac{dk_j}{dt}=-k_i\frac{\partial v_j}{\partial x^j}, \tag{6.3}$$

where the time derivative is taken along the ray:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{U}\cdot\boldsymbol{\nabla}. \tag{6.4}$$

The evolution of the amplitude a_0 is linked with that of the relative energy density W_r, through

$$\langle W_r\rangle=\frac{1}{2}a_0^2\rho_{(0)}\frac{\omega_r^2}{c^2}. \tag{6.5}$$

For a homogeneous stationary fluid we would expect our macroscopic plane wave to correspond to a quantum coherent state whose energy is, in terms of the (quantum) average phonon density $\bar{N}$ and total volume V,

$$E_{tot}=V\langle W_r\rangle=V\bar{N}\hbar\omega_r . \tag{6.6}$$

Since it is a density of ''particles,'' $\bar{N}$ should remain the same when viewed from any frame. Consequently, the relation

$$\bar{N}\hbar=\frac{\langle W_r\rangle}{\omega_r} \tag{6.7}$$

should hold true generally. In classical fluid mechanics the quantity $\langle W_r\rangle/\omega_r$ is called the *wave action* [9,11,13].

The time averages of other components of the energy-momentum tensor may also be expressed in terms of $\bar{N}$. For the mixed tensor we find

$$\langle T_0^0\rangle=\langle W_r+\mathbf{v}\cdot\rho_{(1)}\mathbf{v}_{(1)}\rangle=\bar{N}\hbar\omega,$$

$$\langle T_0^i\rangle=\left\langle\left(\frac{P_{(1)}}{\rho_{(0)}}\right)+\mathbf{v}\cdot\mathbf{v}_{(1)}\right\rangle(\rho_{(0)}v_{(1)i}+\rho_{(1)}v_i)=\bar{N}\hbar\omega U_i ,$$

$$\langle -T_i^0\rangle=\langle\rho_{(1)}v_{(1)i}\rangle=\bar{N}\hbar k_i ,$$

$$\langle -T^i_j\rangle = \langle \rho_{(0)} v_{(1)i} v_{(1)j} + v_i \rho_{(1)} v_{(1)j} + \delta_{ij} P_{(2)}\rangle = \overline{N}\hbar k_j U_i . \tag{6.8}$$

In the last equality we have used that $\langle P_{(2)}\rangle = 0$ for a plane progressive wave.

If we insert these expressions for the time averages into the Blokhintsev energy conservation law (5.27), we find that

$$\frac{\partial \overline{N}\hbar\omega}{\partial t} + \nabla\cdot(\overline{N}\hbar\omega\mathbf{U}) = 0. \tag{6.9}$$

We can write this as

$$\overline{N}\hbar\left(\frac{\partial\omega}{\partial t} + \mathbf{U}\cdot\nabla\omega\right) + \hbar\omega\left(\frac{\partial\overline{N}}{\partial t} + \nabla\cdot(\overline{N}\mathbf{U})\right) = 0. \tag{6.10}$$

The first term is proportional to $d\omega/dt$ taken along the rays and vanishes for a steady mean flow as a consequence of the Hamiltonian nature of the ray tracing equations. The second term must therefore also vanish. This vanishing represents the conservation of phonons, or, in classical language, the conservation of wave action. Conservation of wave action is an analogue of the adiabatic invariance of E/ω in the time-dependent harmonic oscillator.

In a similar manner, the time average of Eq. (5.28) may be written

$$\begin{aligned} 0 &= \frac{\partial \overline{N} k_j}{\partial t} + \nabla\cdot(\overline{N}k_j\mathbf{U}) + \overline{N}k_i \frac{\partial v_i}{\partial x^j} \\ &= \overline{N}\left(\frac{\partial k_j}{\partial t} + \mathbf{U}\cdot\nabla k_j + k_i\frac{\partial v_i}{\partial x^j}\right) + k_j\left(\frac{\partial \overline{N}}{\partial t} + \nabla\cdot(\overline{N}\mathbf{U})\right). \end{aligned} \tag{6.11}$$

We see that the momentum law becomes equivalent to phonon-number conservation combined with the ray tracing equation (A16).

VII. DISCUSSION

The possibility of interpreting the time average of the momentum conservation law in terms of quantum quasiparticles warns us that we are dealing with pseudomomentum and not with Newtonian momentum [18]. Nonetheless, the quantity $\langle \rho_{(1)}\mathbf{v}_{(1)}\rangle = \overline{N}\hbar\mathbf{k}$ is reliably computed from the linearized wave equation and is *part* of the true momentum. It is simply not all of it. Even in the absence of a mean flow with its $\langle \rho_{(2)}\mathbf{v}\rangle$ contribution we still have to contend with $\rho_{(0)}\langle \mathbf{v}_{(2)}\rangle$, and this can be important. As an example [18], consider a closed cylinder filled with fluid. At one end of the cylinder a piston is driven so as to generate plane sound waves which completely span the cross section of the tube. At the other end a second piston is driven at the same frequency with its phase adjusted so as to absorb the sound waves without reflection. It easy to see that an extra pressure equal to $\langle W_r\rangle$ is exerted on the ends of the tube over and above whatever isotropic pressure acts on the ends and sides equally. It is ''obvious'' that this is the force per unit area $\overline{N}\hbar\mathbf{k}c$ required to generate and absorb the phonon beam ''momentum.'' Unfortunately for this simple idea, it is equally obvious that the time average center-of-mass velocity of the fluid in the tube vanishes, so the true momentum density in the beam is exactly zero. The $\langle \rho_{(1)}\mathbf{v}_{(1)}\rangle$ contribution to the momentum density is exactly cancelled by a $\rho_{(0)}\langle \mathbf{v}_2\rangle$ counterflow. This Eulerian streaming is driven by the fluid source term for $\langle \mathbf{v}_{(2)}\rangle$ implicit in Eq. (5.16) [32]. (In a Lagrangian description the particles merely oscillate back and forth with no secular drift.) The momentum *flux* however is exactly the same *as if* (the italics are from [18]) there were no medium and the phonons were particles possessing momentum $\hbar\mathbf{k}$. This is frequently true: the flux of pseudomomentum is often equal to the flux of true momentum to $O(a^2)$ accuracy. Pseudomomentum flux can therefore be used to compute forces. On the other hand, the density of true momentum in the fluid and the density of pseudomomentum are usually unrelated.[2]

It should be said that the $\rho_{(0)}\langle \mathbf{v}_2\rangle$ counterflow will not always cancel the $\rho_{(1)}\langle \mathbf{v}_1\rangle$ wave pseudomomentum [35]. The $\langle \mathbf{v}_2\rangle$ flow depends the geometry. It is found from the source equation (5.16) and from the force the sound field applies to the fluid. The latter will be small when there is no dissipation, as is the case in a superfluid, and for an isolated sound beam source in an infinite medium $\langle \mathbf{v}_2\rangle$ will consist of a flow directed radially inward toward the transducer of sufficient magnitude to supply the mass flowing out along the sound beam [32]. In the presence of dissipation the force becomes important, leading to acoustic streaming.

Consider our closed cylinder further. From Eq. (4.13) we see that in a system with fixed $\langle P\rangle$, and in the presence of the sound wave, the mean density of the fluid will be reduced by

$$\langle \rho_{(2)}\rangle = -\frac{\langle W_r\rangle}{c^2}\left.\frac{d\ln c}{d\ln\rho}\right|_{\rho_{(0)}}. \tag{7.1}$$

Since our cylinder has fixed volume, this density reduction cannot take place. Instead, it is opposed by a pressure on the cylinder wall

$$\Delta P = \langle W_r\rangle \left.\frac{d\ln c}{d\ln\rho}\right|_{\rho_{(0)}}, \tag{7.2}$$

which must be added to the isotropic pressure in the absence of the sound wave. The complete radiation stress tensor is therefore

$$\langle \Sigma_{ij}\rangle = \langle W_r\rangle\left(\frac{k_i k_j}{k^2} + \delta_{ij}\frac{d\ln c}{d\ln\rho}\right). \tag{7.3}$$

This result goes back to Brillouin [34]. The true radiation stress therefore differs from the pseudomomentum flux tensor in its isotropic part. Forces computed from pseudomo-

[2]This does not mean that the attribution of momentum to a phonon in the two-fluid model for a superfluid is incorrect. In superfluid hydrodynamics the $\rho_{(0)}\langle \mathbf{v}_2\rangle$ counterflow is accounted for separately from the $\langle \rho_{(1)}\mathbf{v}_{(1)}\rangle = \overline{N}\hbar\overline{\mathbf{k}}$ normal component mass flux. The counterflow is included in the supercurrent needed to enforce $\nabla\cdot(\rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s) = 0$.

mentum flux will therefore be incorrect when this pressure is important. See [36] for examples.

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APPENDIX: GEODESICS AND HAMILTONIAN FLOWS

In this appendix we show that the null geodesics of the Unruh metric coincide with conventional Hamiltonian optics ray tracing. The ray tracing equations are derived from $\omega(\mathbf{k},\mathbf{x})$ as

$$\dot{\mathbf{x}}=\frac{\partial\omega}{\partial\mathbf{k}},\quad \dot{\mathbf{k}}=-\frac{\partial\omega}{\partial\mathbf{x}}. \tag{A1}$$

In our case $\omega(\mathbf{k},\mathbf{x})=c|k|+\mathbf{v}\cdot\mathbf{k}$. Thus

$$\frac{dx^i}{dt}=v_i+c\frac{k_i}{|k|},\quad \frac{dk_i}{dt}=-\frac{\partial v_j}{\partial x^i}k_j. \tag{A2}$$

We begin by noting that geodesics with an affine parameter τ are stationary paths for the Lagrangian

$$L=\frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{A3}$$

To make connection with the ray tracing formalism, consider the corresponding Hamiltonian

$$H=\tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu, \tag{A4}$$

and write down Hamilton's equations with τ playing the role of time:

$$\frac{dx^\mu}{d\tau}=\frac{\partial H}{\partial p_\mu}=g^{\mu\nu}p_\nu,$$

$$\frac{dp_\mu}{d\tau}=-\frac{\partial H}{\partial x^\mu}=-\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}p_\alpha p_\beta. \tag{A5}$$

Combining these gives

$$\frac{d^2x^\mu}{d\tau^2}=\frac{\partial g^{\mu\beta}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}p_\mu+g^{\mu\nu}\left(-\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\nu}\right)p_\alpha p_\beta. \tag{A6}$$

To see that this is the geodesic equation, note that

$$d\mathbf{g}^{-1}=-\mathbf{g}^{-1}(d\mathbf{g})\mathbf{g}^{-1}, \tag{A7}$$

so, with $(\mathbf{g})_{\alpha\beta}=g_{\alpha\beta}$ and $(\mathbf{g}^{-1})_{\alpha\beta}=g^{\alpha\beta}$, we can write

$$\frac{d^2x^\mu}{d\tau^2}+\frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta}+\frac{\partial g_{\nu\beta}}{\partial x^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0. \tag{A8}$$

We now examine these equations for the particular case of the Unruh metric. We define a four-vector $p_\mu=(\omega,-k_i)$ so that $p_\mu x^\mu=\omega t-\mathbf{k}\cdot\mathbf{x}$. Then

$$H=\tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu=\tfrac{1}{2}[(\omega-\mathbf{v}\cdot\mathbf{k})^2-c^2|k|^2]. \tag{A9}$$

Hamilton's equations become

$$\frac{dx^0}{d\tau}=\frac{dt}{d\tau}=\frac{\partial H}{\partial\omega}=\omega-\mathbf{v}\cdot\mathbf{k} \tag{A10}$$

and

$$\frac{dx^i}{d\tau}=-\frac{\partial H}{\partial k_i}=v_i(\omega-\mathbf{v}\cdot\mathbf{k})+c^2k_i. \tag{A11}$$

For null geodesics $(\omega-\mathbf{v}\cdot\mathbf{k})^2-c^2|k|^2=0$, or $(\omega-\mathbf{v}\cdot\mathbf{k})=c|k|$. Thus

$$\frac{dx^i}{dt}=v_i+\frac{c^2k_i}{(\omega-\mathbf{v}\cdot\mathbf{k})}, \tag{A12}$$

or

$$\frac{dx^i}{dt}=v_i+c\frac{k_i}{|k|}, \tag{A13}$$

which is the group velocity equation. We also find

$$\frac{d\omega}{d\tau}=-\frac{\partial H}{\partial t}=0 \tag{A14}$$

if the flow is steady, and

$$-\frac{dk_i}{d\tau}=-\frac{\partial H}{\partial x^i}=(\omega-\mathbf{v}\cdot\mathbf{k})\frac{\partial v_j}{\partial x^i}k_j, \tag{A15}$$

which is equivalent to the momentum evolution equation

$$\frac{dk_i}{dt}=-\frac{\partial v_j}{\partial x^i}k_j. \tag{A16}$$

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